V.N. Karazin Kharkiv National University
B. Verkin Institute for Low Temperature Physics and Engineering
of the National Academy of Sciences of Ukraine
Kharkiv Mathematical Society

International Conference

Modern Advances in Geometry and Topology

in honor of professor A.A. Borisenko
for his 70th birthday

September 12–16, 2015

Book of Abstracts

Kharkiv 2016

International Conference "Modern Advances in Geometry and Topology": Book of abstracts, Kharkiv: V.N. Karazin Kharkiv National University, 2016, 52 p.

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Sponsored by

The Commission for Developing Countries of the International Mathematical Union (Conference Support Program)

and the private donation of Aleksey A. Borisenko

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Modern Advances in Geometry and Topology 2016

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Alexander Andreevich Borisenko was born on May 24, 1946 in Lebedin town (Sumy region, Ukraine). In 1969 he graduated from the Mechanics and Mathematics Department of Kharkiv State University and enrolled in the geometry graduate school. In 1973 he defended his PhD thesis, "The structure of surfaces with degenerate spherical image" (Kharkiv State University), and in 1983 he defended his habilitation thesis "Multi-dimensional surfaces of nonpositive extrinsic curvature" (Moscow State University). In 1995 he was elected as a corresponding member of the NAS of Ukraine. In 2002 A. Borisenko was awarded the Krylov Prize of the National Academy of Sciences of Ukraine, followed by the State Prize of Ukraine in the field of science and technology in 2005 and A.V. Pogorelov Prize of National Academy of Sciences of Ukraine in 2010.

Since 1973 he worked at the Department of Geometry of the Kharkiv State University, and was the Department Chair from 1980 by 2012. During this period the Kharkiv geometrical school got a new impulse in development of the geometry of multidimensional submanifolds in Euclidean, Riemannian, pseudo-Riemannian, Finsler spaces and spaces with additional structures (complex, Sasakian, fiber etc.). His research interests are very much "multivariate". To prove this it is enough to look at the variety of his post-graduate students and collaborators: D. Bolotov (geometry and topology of foliations), K. Drach (isoperimetric inequalities, comparison theorems, the optimal control), N. Farafonova (geometry of Grassmann bundles), V. Lisitsa (geometry of helical submanifolds), O. Lykova (geometry of the Grassmann image of complex submanifolds), V. Miquel (curvature flows, geometry of convex hypersurfaces in Hadamard manifolds), Y. Nikolayevsky (geometry of the Grassmann image of submanifolds), S. Okrut (geometry of distributions on manifolds), Ye. Olin (Finsler geometry), E. Petrov (geometry of submanifolds in Lie groups), L. Sergienko (nonholonomic geometry), K. Tenenblat (Minkowski geometry), V. Ushakov (geometry of ruled submanifolds), A. Yampolsky (geometry of fiber bundles). From 2012 by 2015 Alexander Borisenko was a professor of the department of mathematical analysis and optimization methods of Sumy State University. Since 2016 he is a leading researcher at the Mathematical Department of B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine.

As a visiting professor, he worked in the University of Haifa (Israel, 2013, 2010), Technical University of Berlin (Germany, 2010), University of Rome "La Sapienza" (Italy, 2003), Autonomous University of Barcelona (Spain, 2000), Centre de Recerca Mathematica (Barcelona, 2000), University of Valencia (Spain, 1997, 1999, 2000, 2004, 2008), University of Brasilia (Brazil, 1996, 2008), Bilkent University (Turkey, 1995). A. Borisenko took part as a Speaker in the International Congress of Mathematicians in Beijing (2002) and Zurich (1994).

Alexander Borisenko is the author of more than 120 articles and two monographs (Intrinsic and extrinsic geometry of multidimensional manifolds / A. Borisenko - M.: Examen, 2003. - 670 p.; An introduction to Hamilton and Perelman's work on the conjectures of Poincare and Thurston / A. Borisenko, E. Cabezas-Rivas, V. Miquel-Molina. - Matematiques, 2006. - 3 (2) - 150 p.). He is the author of textbooks in Analytical Geometry (1993) and Differential Geometry and Topology (1995) for undergraduates.

To Hopf's conjecture about metric on the topological product $S^2 \times S^2$ of two 2-spheres

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Hopf's well-known conjecture states that there exists no metric of stricly positive curvature on the topological product $S^2 \times S^2$ of two 2-spheres.

Note that, by Preissmann's theorem [1], on the topological product $M \times N$ of two compact differentiable manifolds, there exists no metric of strictly negative curvature.

In [2] M.Berger showed that if the sectional curvature K of a metric on the $S^2 \times S^2$ satisfies the inequalities $\delta \leq K \leq 1$, then $\delta < \frac{4}{17}$.

At our talk we expose three theorems from our article [3].

We consider the minimal cycles F_1^2 and F_2^2,which generate group $\pi(S^2 \times S^2)$. Let P_0 is its point of intersection. Suppose that the metric on $S^2 \times S^2$ has the form

$$ds^2 = \sum_{i,j=1}^{4} g_{ij} du^i du^j.$$

Let minimal cycles are coordinate surfaces :$F_1^2 : u^3 = 0, u^4 = 0$, $F_2^2 : u^1 = 0, u^2 = 0$. We assume that $g_{13} = g_{14} = g_{23} = g_{24} = 0$ and $g_{12} = 0$ on F_1^2 , $g_{34} = 0$ on F_2^2. (1)

Definition. We say that a globally minimal surface F_1^2 is **uniformly stable** in the family of surfaces $u^3 = c^3, u^4 = c^4$ if the stability condition $\frac{\partial^2 \sqrt{g_{11}g_{22} - g_{12}^2}}{\partial u^i \partial u^i} \geq 0, i = 3, 4$ holds at each point of this surface.

Theorem 1. *If globally minimal cycles in the product $M = S^2 \times S^2$ are uniformly stable and the metric satisfies the condition (1), then , at the orthogonal intersection point of the cycles, the curvature of M is non positive, at least for some area element.*

Theorem 2. *If , on the topological product $M = S^2 \times S^2$ with a Riemannian metric , there exist orthogonal coordinates $u^1, ..., u^4$ and all 2-dimensional coordinate surfaces $u^1 = const, u^2 = const$ and $u^3 = const, u^4 = const$ are minimal, then the integral inequality*

$$\int_M (K_{13} + K_{23} + K_{14} + K_{24}) dV \leq 0$$

holds, where dV is the volume element of M and K_{ij} is the curvature of coordinate surface.

Consider metric on $M = S_1^2 \times S_2^2$ of the form

$$ds^2 = \Phi_1(P_1, P_2)ds_1^2 + \Phi_2(P_1, P_2)ds_2^2, \tag{2}$$

where P_i and ds_i^2 - a point and metric on S_i^2. Let $\Phi_i(P_1, P_2)$ be C^2 regular functions, which are positive at all values of their arguments and have the form

$$\Phi_1 = \sum_k^p A_k(P_1)B_k(P_2), \quad \Phi_2 = \sum_k^q C_k(P_1)D_k(P_2). \tag{3}$$

Let all functions $B_k(P_2)$ have a minimum at same point P_{20} and all functions $C_k(P_1)$ have a minimum at same point P_{10}.Also suppose that $A_k \geq 0, D_k \geq 0.$ (4)

Theorem 3. *Metric (2) on the topological product $S^2 \times S^2$ under assumptions (3-4) has non positive curvature at some point and for some tangent area element.*

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Almost geodesic mappings of first type on spaces with affine connection onto symmetric spaces

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N.S. Sinyukov defined *almost geodesic mappings* f: $A_n \to \bar{A}_n$ of spaces with affine connection [1]. He defined three types of these mappings: π_1, π_2 and π_3. We proved [2] that for dimension $n > 5$ there exist precisely these three types. See [3, p. 455-480].

It is known [1] that any mapping π_1 can be expessed as a composition of a geodesic mapping and a canonical mapping π_1. *Canonical* almost geodesic mappings of first type are characterized by the following equations [3, p. 464]:

$$3(P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}) = R^h_{(ij)k} - \bar{R}^h_{(ij)k} + \delta^h_{(k}a_{ij)}, \tag{1}$$

where P^h_{ij} is a deformation tensor of the connections, R^h_{ijk} and $\bar{R}^h_{ijk}$ are the curvature tensors of A_n and $\bar{A}_n$, δ^h_i is the Kronecker delta, a_{ij} is a symmetric tensor, round parentheses denote a symmetrization of indices without division by , and a comma denotes the covariant derivative on A_n.

Spaces with affine connection are called *symmetric* if their curvature tensor is absolutely parallel, see [3, p. 87].

If a mapping $f\colon\ A_n \to \bar{A}_n$ and $\bar{A}_n$ is symmetric, then in A_n the following conditions are satisfied:

$$\bar{R}^h_{ijk,m} = -P^h_{m\alpha}\bar{R}^\alpha_{ijk} + P^\alpha_{mi}\bar{R}^h_{\alpha jk} + P^\alpha_{mj}\bar{R}^h_{i\alpha k} + P^\alpha_{mk}\bar{R}^h_{ij\alpha}. \tag{2}$$

When studying the integrability conditions of equations (1) we found:

$$\begin{gathered}(n-1)\,a_{ij,k} = -3\,P^\beta_{\alpha(i}R^\alpha_{j)k\beta} - P^\beta_{\alpha k}R^\alpha_{(ij)\beta} + P^\beta_{\alpha\beta}R^\alpha_{(ij)k} + R^\beta_{(ij)k,\beta} - \\ R_{(ij),k} + 3\,P^\alpha_{ij}\bar{R}_{\alpha k} - P^\beta_{\alpha i}\bar{R}^\alpha_{(j\beta)k} + P^\beta_{ki}\bar{R}_{(j\beta)} - P^\beta_{\alpha j}\bar{R}^\alpha_{(i\beta)k} + \\ P^\beta_{kj}\bar{R}_{(i\beta)} - \delta^\alpha_{(\beta}a_{ij)}P^\beta_{\alpha k} + \delta^\alpha_{(k}a_{ij)}P^\beta_{\alpha\beta} - \tfrac{1}{n+2}\,B_{(ij)k},\end{gathered} \tag{3}$$

where

$$\begin{gathered}B_{ijk} = P^\beta_{\alpha k}(R^\alpha_{ij\beta} + R^\alpha_{\beta ji}) - P^\beta_{\alpha j}(R^\alpha_{ik\beta} + 3\,R^\alpha_{\beta ki}) + 3\,P^\beta_{\alpha\beta}R^\alpha_{ijk} + 3\,R^\alpha_{ijk,\alpha} - \\ R_{(ij),k} + R_{(ik),j} + 2\,P^\alpha_{ij}\bar{R}_{\alpha k} - 2\,P^\alpha_{ik}\bar{R}_{\alpha j} + P^\alpha_{ki}\bar{R}_{j\alpha} - P^\alpha_{ij}\bar{R}_{k\alpha} - \\ P^\beta_{\alpha j}\bar{R}^\alpha_{(i\beta)k} + P^\beta_{\alpha k}\bar{R}^\alpha_{(i\beta)j} - a_{ij}P^\alpha_{\alpha k} - a_{\alpha j}P^\alpha_{ik} + a_{ik}P^\alpha_{\alpha j} + a_{\alpha k}P^\alpha_{ij},\end{gathered}$$

and R_{ij} and $\bar{R}_{ij}$ are the Ricci tensors on A_n and $\bar{A}_n$, respectively.

The equations (1), (2) and (3) are a closed system of PDE with covariant derivatives of Cauchy type with respect to the unknown functions P^h_{ij}, a_{ij} and $\bar{R}^h_{ijk}$, which, naturally, have to satisfy the following algebraic conditions

$$P^h_{ij}(x) = P^h_{ji}(x), \quad a_{ij}(x) = a_{ji}(x), \quad \bar{R}^h_{i(jk)}(x) = \bar{R}^h_{(ijk)}(x) = 0. \tag{4}$$

We proved

Theorem *A space A_n with affine connection admits canonical almost geodesic mappings of first type onto symmetric spaces $\bar{A}_n$ if and only if in A_n the equations (1), (2), (3) and (4) have a solution with respect to the functions P^h_{ij}, a_{ij} and $\bar{R}^h_{ijk}$.*

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Quasiareal deformation in class of surfaces of constant mean curvature

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Quasiareal deformation is understood as an infinitesimal deformation of the first order with the given law of changing the element of area of a surface in Euclidean three-space.

Let $\overline{U}\left(x^1,x^2\right)$ be a field of velocities of the points of the surface $\overline{r} = \overline{r}\left(x^1,x^2\right)$ at the initial moment of the deformation, such that $\overline{U} = U^\alpha \overline{r}_\alpha + U^0\overline{n}$, where $\overline{r_i}, \overline{n}$, $i = 1, 2$, are the basis vectors. The fundamental equations of the quasiareal infinitesimal deformation, which are expressed in terms of the components of the partial derivatives of the field $\overline{U}$, are derived in [2].

In this paper it has been established: in order that the field $\overline{U} \in C^1$ be a deforming field of the quasiareal infinitesimal deformation it is necessary and sufficient that the components U^α, U^0 satisfy the equation

$$U^\alpha_{,\alpha} - 2HU^0 = -2\mu, \tag{1}$$

where the function μ expresses the law of changing the element of area.

It is evident, that the class of the quasiareal infinitesimal deformation is very wide since one differential equation (1) contains four unknown functions. It is expedient to study such deformation under the additional geometrical or mechanical conditions. For example, for the surface of constant mean curvature on the condition that $\delta H = 0$ under the quasiareal infinitesimal deformation we have additional elliptic partial differential equation of the second order with respect to the normal component of the deforming field

$$g^{\alpha\beta}U^0_{\alpha,\beta} + 2\left(2H^2 - K\right)U^0 = 0. \tag{2}$$

The Riemann domain T has been described, in which the regular solution of the equation (2) exists for the regular surfaces of constant mean curvature, this solution is a continuous, non-zero everywhere in closed domain $\overline{T}$. This condition is a sufficient sign of the existence and uniqueness of the solution of the Dirichlet problem for the equation (2) [1].

The corresponding theorems have been formulated for the quasiareal infinitesimal deformation in the class of the surfaces of constant mean curvature. An infinitesimal deformation of the first order of the surfaces of constant mean curvature is discussed, for example, in a paper [3].

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Isometricity of pretangent spaces to convex subsets of Euclidean spaces

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A pretangent space to the metric space (X,d) at a point p is a set of equivalence classes of sequences $\tilde{x} = (x_n)_{n\in\mathbb{N}}$ of points from X, tending to a sequence $(p, p, ...)$ with a given rate, [6]. We will discuss the problem of description the conditions under which all pretangent spaces to an arbitrary metric space X at a point p are isometric to convex sets in n-dimensional Euclidean space E^n. By D. Dordovskyi in [3] it was proved that, for all points in E^n, every pretangent space to E^n is isometric to E^n. Moreover, pretangent spaces to convex and star-like sets on the Euclidean plane have been completely described by O. Dovgoshey, F. Abdullayev and M. Küçükaslan in [4] and [5].

We consider the concept of midpoint convexity of metric space X at a point that give us conditions of geodesity of pretangent spaces, [1]. Recall that a metric space X is midpoint convex at a point $p \in X$ if for every two tending to p sequences $\tilde{x}$ and $\tilde{y}$ there exists a sequence $\tilde{z}(\tilde{x}, \tilde{y})$, such that $d(x_n, z_n) = \frac{1}{2}d(x_n, y_n) + o(\max\{d(x_n, p), d(y_n, p)\})$, $d(y_n, z_n) = \frac{1}{2}d(x_n, y_n) + o(\max\{d(x_n, p), d(y_n, p)\})$ and $\frac{|d(x_n,z_n)-\frac{1}{2}d(x_n,y_n)|}{\max\{d(x_n,p),d(y_n,p)\}} = \frac{|d(y_n,z_n)-\frac{1}{2}d(x_n,y_n)|}{\max\{d(x_n,p),d(y_n,p)\}} = 0$. We also use the infinitesimal versions of the classical embedding theorems in E^n which were obtained in [2]. In conclusion, the simple idea that a metric space Y is isometric to a convex subset of E^n if and only if Y is geodesic and isometrically embedded in E^n leads to the solving of our problem.

This work was supported by the grant of the State Fund for Fundamental Research (Project N 20570) and by Project 15-1bb$\backslash 19$ "Metric Spaces, Harmonic Analysis of Functions and Operators and Singular and Nonclassic Problems for Differential Equations" (Donetsk National University, Vinnitsia, Ukraine).

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Foliations of nonnegative curvature

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We are going to discuss our recent results about codimension one foliations of nonnegative curvature on closed manifolds (see [1], [2]). In particular:

- we prove the Milnor conjecture for the leaves of codimension one nonnegative Ricci foliations on closed manifolds and we show that the fundamental group of such manifolds should be almost poly-ciclic;
- we establish the criteria of flatness of Ricci foliations, in particular, we show that the foliation is flat iff the manifold is of homotopy type $K(\pi, 1)$;
- we give the affirmative answer to the G. Stuck's question about nonexistence of codimension one foliations of nonnegative sectional curvature on spheres except S^3.

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Incidence axioms for the boundary at infinity of complex hyperbolic spaces

Sergei Buyalo[1], Viktor Schroeder[2]

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[2] *Institut für Mathematik, Universität Zürich, Germany*

We characterize the boundary at infinity of a complex hyperbolic space as a compact Ptolemy space that satisfies four incidence axioms.

Extreme problems for curves and surfaces with bounded curvature

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A hypersurface is called λ-convex (resp., λ-concave), if it is locally convex and at each point there exists an outward (resp., inward) supporting totally umbilical hypersurface of curvature equal to λ. In the smooth case the condition of being λ-convex (concave) is equivalent to the boundedness of normal curvatures as $k_n \geqslant \lambda$ (resp., $\lambda \geqslant k_n \geqslant 0$) at each point and in every direction.

In the talk we will consider two series of problems. The first series is about a so-called reverse isoperimetric problem, that is minimization of area (volume) of compact domains assuming that the lengths (surface areas) of the boundaries are given and fixed. For curves in $\mathbb{R}^2$ with bounded absolute curvature the problem was addressed in [4]. For λ-convex curves on planes of constant curvature – in [1], [2], [3]. We extend the results of Borisenko and Drach for λ-concave curves on the Euclidean plane and in the 2-dimensional de Sitter space. Moreover, we prove the reverse isoperimetric inequality in the 3-dimensional Euclidean space for λ-concave surfaces.

The second series of questions we will cover in the talk is dealing with minimization of area (volume) of a bounded domain with a given diameter. A hypersurface is called λ_1, λ_2-convex, if it is λ_1-convex and λ_2-concave. For λ_1, λ_2-convex hypersurface in $\mathbb{R}^n$ (resp., spherical space $\mathbb{S}^n$ and hyperbolic space $\mathbb{H}^n$) we prove the reverse isodiametric inequality which states that only a so-called spindle-shaped hypersurface encloses a minimal volume among all λ_1, λ_2-convex of given diameter.

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Minimal covers of Archimedean toroids

K.D. Drach[1], Y.V. Haidamaka[1]

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The talk focuses on toroidal maps of certain type and their covers. We say that a finite graph X embedded in a torus S such that every connected component of $S\backslash X$ is homeomorphic to an open disc is called a *map* on the torus S.

A map M obtained as a quotient of an Archimedean tessellation τ by some translation subgroup G generated by two linearly independent vectors and which preserves τ called an *Archimedean toroidal map (toroid)*. Here an *Archimedean tessellation* is a tessellation of the Euclidean plane by regular polygons of two or more types, such that any two vertices can be mapped into each other by some symmetry of τ.

Finally, we call a toroidal map M *almost regular* if it has a minimal number of flag orbits under the action of the automorphism group of M. Note that for Archimedean toroids this number is never equal to 1, that is Archimedean toroids are never *regular*.

The main result of the talk is summarized in the following theorem.

Theorem 1. *Each Archimedean map on the torus has a unique minimal almost regular toroidal cover. Moreover, this cover can be constructed explicitly.*

This theorem extends the results obtained in [1] for minimal regular and rotary covers of equaivelar toroidal maps. In the talk we will give all necessary definitions and outline the idea of the proof.

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Scissors-congruence for unbounded polygons and polyhedra

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Two polytopes A and B are called *scissors-congruent* ($A \sim B$) if they can be dissected into equal finite sets of polytopal pieces. The classical Bolyai–Gerwien theorem states that two polygons are scissors-congruent if and only if they have equal areas [2]. However, for polyhedra equality of their volumes is not enough for being scissors-congruent. For 3-polytopes the Dehn–Sidler theorem states that two bounded polyhedra are scissors-congruent if and only if their volumes and so-called Dehn's invariants are equal [2].

We address the question of scissors-congruence for unbounded polyhedra in $\mathbb{R}^2$ and $\mathbb{R}^3$.

A polygon is said to be *unbounded* if its boundary is composed of finite number of straight line segments and half-lines. A polyhedron is called *unbounded* if it has unbounded faces.

For an unbounded polytope (polygon) A take an arbitrary point O and draw from it all half-lines parallel to the half-lines lying in A (and having the same directions). These half-lines fill in some conical surface with vertex O. We call this surface the *limit angle* $\phi(A)$ of A [1]. Denote by $|\phi(A)|$ the angular measure of $\phi(A)$.

If all unbounded parts of A consist of parallel half-line edges, the limit angle reduced to a set of half-lines and $|\phi(A)| = 0$.

Theorem 1. *Let A and B be unbounded polygons in $\mathbb{R}^2$.*

1. *If $|\phi(A)| + |\phi(B)| \neq 0$, then $A \sim B$ if and only if $|\phi(A)| = |\phi(B)|$.*
2. *If $|\phi(A)| + |\phi(B)| = 0$, then $A \sim B$ if and only if they have equal sum of distances between the parallel half-line edges in each unbounded part.*

Let $A \subset \mathbb{R}^3$ be an unbounded polyhedron such that $\phi(A)$ is the set of rays, i.e. all unbounded parts are prisms P_i, $i = \overline{1,k}$. For each P_i draw a cross section π_i not intersecting the half-line edges of A and orthogonal to the edges of P_i. Let P_i' be an infinite part of P_i bounded by π_i. Thus A is naturally decomposed into *finite* A^f and *infinite* A^∞ parts, where $A^\infty = \bigcup_{i=1}^k P_i'$ and $A^f = \overline{A \setminus A^\infty}$. Denote by $W(A)$ the sum of the areas of the bases of P_i'.

Theorem 2. *Two unbounded polyhedra $A, B \subset \mathbb{R}^3$ whose limit angles degenerate into sets of half-lines are scissors-congruent if and only if $W(A) = W(B)$ and $A^f \cup P \sim B^f \cup Q$ for some cubes P and Q.*

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Fundamental groups of orbits of smooth functions on 2-torus

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Let M be a smooth surface, and $\mathcal{D}(M)$ be the group of diffeomorphisms of M. The group $\mathcal{D}(M)$ acts on the space $C^\infty(M)$ of smooth functions on M by the following rule:

$$\gamma : C^\infty(M) \times \mathcal{D}(M) \to C^\infty(M), \qquad \gamma(f, h) = f \circ h.$$

Under the action γ we will define the orbit

$$\mathcal{O}(f) = \{f \circ h \,|\, h \in \mathcal{D}(M)\}$$

of $f \in C^\infty(M)$. Endow the space $C^\infty(M)$ with the corresponding Whitney topology. This topology induces some topology on $\mathcal{O}(f)$. Let also $\mathcal{O}_f(f)$ be a connected component of $\mathcal{O}(f)$, which contains f.

Let $\mathcal{F}(M) \subset C^\infty(M)$ be the set of smooth functions satisfying the following two conditions:

(B) the function f takes a constant value at each connected component of ∂M, and all critical points of f belong to the interior of M;

(P) for each critical point x of f the germ (f, x) of f at x is smoothly equivalent to some homogeneous polynomial $f_x : \mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

For the function $f \in \mathcal{F}(T^2)$ on 2-torus T^2 we obtain the description of $\pi_1\mathcal{O}_f(f)$, see [1]–[5].

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On one way of finding of infinitesimal bendings of convex surfaces with boundary conditions in conformally Euclidean space

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n.1. Let R^3 be a three dimensional conformally Euclidean space with metric $ds^2 = E(Z)(dX^2 + dY^2 + dZ^2)$, where $E \in C^{4,v}$, $0 < v < 1$, $E(Z) > 0$.

n.2. Let further S, $S \in R^3$, be a simply-connected surface, whose boundary is a closed smooth curve. We suppose that S is given be equations $X = x$, $Y = y$, $Z = f(x, y)$, $(x, y) \in D$, where D is a domain of the Oxy – plane, $f \in C^{3,v}$, $(0, 0, f(0, 0))$ is a reinforced umbilical point of S.

n.3. Let $\{\xi, \eta, \zeta\}$ be the components of the displacement field of the surface S. Then the functions ξ, η, ζ satisfy the following differential equations

$$\left.\begin{aligned} \xi_x + p\zeta_x + (\ln\sqrt{E})'(1 + p^2)\zeta = 0;\\ \xi_y + \eta_x + p\zeta_y + q\zeta_x + 2(\ln\sqrt{E})'pq\zeta = 0;\\ \eta_y + q\zeta_y + (\ln\sqrt{E})'(1 + q^2)\zeta = 0, \end{aligned}\right\} \tag{1}$$

where $(x, y) \in D$, $p = f_x$, $q = f_y$; index "$'$" means the derivative by Z of the function $\ln\sqrt{E}$.

n.4. Let us consider a field $\{x \sin\gamma, y \sin\gamma, r \cos\gamma\}$ on the boundary ∂S, where γ is a given function of class $C^{1,v}$, $r^2 = x^2 + y^2$. We consider the infinitesimal bendings of the surface S with the boundary condition

$$x \sin\gamma \cdot \xi + y \sin\gamma \cdot \eta + r \cos\gamma \cdot \zeta = \sigma\ ,\ (x, y) \in \partial D \tag{2}$$

where σ is a given function on ∂D .

n.5. Necessary and sufficient criteria for the decision problem (1), (2) is given the decision following problem

$$\sum_{i,k=1}^{2} \frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial U}{\partial x_i}\right) + \sum_{i=1}^{2} e_i \frac{\partial U}{\partial x_i} + cU = 0\ ,\ \text{on } D \tag{3}$$

$$r\frac{\partial U}{\partial r} + bU = \sigma_1\ ,\ \text{on } \partial D, \tag{4}$$

where a_{ik}, e_i, c, b, σ_1 are known functions, $x_1 = x$, $x_2 = y$, $a_{11}a_{22} - a_{12}^2 \geq a_0 > 0$, $a_0 = \text{const}$.

If we know the solution of the problem (3), (4), then we find the functions, ξ, η, ζ as the solution of the problem (1), (2).

Generalized circular tractrices and Dini surfaces

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Yu. Aminov and A. Sym in [1] asked for a generalization of the classical theory of Bäcklund transformations of pseudo-spherical surfaces in E^3 to the case of surfaces in E^4. This problem was studied in a series of authors papers, see [2]. We discuss a particular problem concerning a generalization to E^4 of the classical Dini surfaces. A Dini surface is a pseudo-spherical surface in E^3 obtained by a screw rotation of a tractrix. Dini surfaces correspond to one-soliton solutions of the sine-Gordon equation and are characterized by a degeneration of their Bäcklund transformations. In order to construct a Dini type surfaces in E^4, we use the notion of *generalized tractrices* in E^3.

Namely, let γ be an oriented regular space curve in E^3. The end-points of unit segments tangent to γ form a curve $\Gamma \subset E^3$ called a directrix of γ. If Γ belongs to $E^2 \subset E^3$, then γ is called *a generalized tractrix* in E^3. Supposing Γ to be a circle of radius r, we reconstruct a generalized tractrix $\gamma \subset E^3$ whose directrix is Γ. Analytically the problem is reduced to an autonomous system of two ODE for two functions. Qualitative properties of the solution depend on r: if $r \geq 1$, then γ is infinite and attracts to a circle of radius $\sqrt{r^2-1}$; if $r < 1$, then γ is finite. The behavior of γ resembles circular tractrices of Euler [3], hence γ is called *a generalized circular tractrix*.

Viewing E^3 as a subspace in E^4, we apply a particular screw rotation in E^4. Then γ sweeps out a surface $F^2 \subset E^4$ with constant Gauss curvature, which admits a Bäcklund type transformation resulting in the circle Γ. Since F^2 inherits fundamental properties of Dini surfaces in E^3, we propose to consider F^2 as *a generalized Dini surface* in E^4. An interesting open problem is to discover how generalized Dini surfaces in E^4 may be interpreted analytically in the frames of the theory of integrable systems including the theory of the sine-Gordon equation.

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Optimal functions with isolated critical points on the boundary of the surfaces

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Topological classification of functions with isolated critical points on compact manyfold was gotten in [1] and of Morse functions on manyfold with the boundary — in [2].

Let M is smooth compact surface with one component of the boundary ∂M, f — smooth function, defined on this surface and which has no more than one critical point on each level line.

Function $f : M \to \Re$ with isolated critical points, which belong to the single component of the boundary of the surface and also are isolated critical points of restriction $f|_{\partial M}$ of function f to the boundary, we will call *optimal* if it has the least possible number of critical points on defined surface among all such functions.

Theorem 1. Optimal function, defined on surface with the boundary, except of 2–dimentional disk, has exactly three critical points.

Chord diagram of saddle critical level line of function on smooth compact surface with the boundary is the circle with the following elements:

(1) marked points, which are enumerated;

(2) chords, both ends of which are marked points and such that have no common ends;

(3) coloration of arcs, which marked points divide the circle on, into two colors, such that every two neighboring arcs have the different color and after interchanging the colors we get the coloration which equals to the first one.

Chord diagram are *equivalent* if the can replace each other by symmetry or (and) turn, such that save the elements (1)–(3).

We can put relatively clearly the chord diagram into the correspondence to some neighborhood of saddle critical level line of function f.

Smooth functions f i g are *topological equivalent* in some neighborhood of their critical level lines $f^{-1}(c_1)$ i $g^{-1}(c_2)$ if there are exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and the homeomorphism $\lambda : f^{-1}(c_1 - \varepsilon_1, c_1 + \varepsilon_1) \to g^{-1}(c_2 - \varepsilon_2, c_2 + \varepsilon_2)$, which transfer level lines of function f into level lines of function g and save the direction of growing of functions.

Theorem 2 (Criterion of topological equivalence). Optimal functions are topologically equivalent iff their chord diagram are equivalent.

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On q^λ and q_0^λ invariant spaces

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Invariant sequence spaces which can be considered as topological sequence spaces are very helpful for investigations of the duality of sequence spaces. For instance, if the sequence space X satisfies the condition $\ell_\infty \cdot X = X$ then its $\alpha-$, β and $\gamma-$ duals (usually known as Köthe- Toeplitz duals) are same [5]. Garling [2] investigated $B-$ and B_0- invariant sequence spaces and Buntinas [3] introduced and investigated $q-$ and q_0- invariant sequence spaces and recently, Grosse- Erdmann [4] studied on ℓ_1 invariant sequence spaces. In this work, we define q^λ and q_0^λ invariant spaces, X with $q^\lambda \cdot X = X$ and $q_0^\lambda \cdot X = X$, respectively.

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Fenchel's Problems for a de Sitter $n-$Simplex

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W. Fenchel raised two questions regarding two sets of $(n(n+1))/2$ positive real numbers in his book [1]. What are the necessary and sufficient conditions for each set to be the dihedral angles and edge lengths of a hyperbolic n-simplex? These problems were solved by Feng Luo ([2]) and Karliga ([3]) by using Gram matrix and Edge matrix of a hyperbolic n-simplex, respectively.

It is natural to pose the above Fenchel's problems and give Gram and Edge matrices of a de Sitter n-simplex. In this paper, we give the necessary and sufficient conditions for a given symmetric matrix to be the Edge or Gram matrix of a de Sitter n-simplex.

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On the geodesic mappings of quasi-Einstein spaces

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We investigate the equation

$$E_{ij} = u_i u_j, \tag{1}$$

where $E_{ij} \overset{def}{=} R_{ji} - \frac{R}{n} g_{ij}$ is the Einstein tensor, R_{ij} is the Ricci tensor, R is the scalar curvature, and u_i is a certain vector of a pseudo-Riemannian space V_n, $n > 2$, with metric tensor g_{ij}. For a long time, Eq. (1) has been attracting the attention of researchers, first of all, due to its application in mechanics, in particular, in fluid mechanics, and also as a generalization of Einstein spaces, i.e., spaces where the Einstein tensor is zero.

Definition 1. The pseudo-Riemannian space V_n different from a space of constant curvature is called a quasi-Einstein space and denoted by M_n if conditions (1) are satisfied in it.

Note that, convolving (1) and taking into account that the Einstein tensor is traceless, one can easily make sure that the vector u_i is necessarily isotropic, i.e.,

$$u_\alpha u^\alpha = 0.$$

Here, $u^i = u_\alpha g^{\alpha i}$, and gy are elements of the matrix inverse to g_{ij}.

Definition 2. A diffeomorphism $f : V_n \to \bar{V_n}$ is called a geodesic mapping of V_n onto $\bar{V_n}$ if f maps any geodesic curve in V_n onto a geodesic curve in $\bar{V_n}$.

We proved following theorems.

Theorem 1. If quasi-Einstein pseudo-Riemannian space V_n has constant scalar curvature and permits trivial geodesic mapping, then it should permit a solution of equations system:

$$\begin{aligned} a_{ij,k} &= \lambda_i g_{jk} + \lambda_j g_{ik} \\ \lambda_{i,\,j} &= \mu g_{ij} + \frac{R}{n(n-1)} a_{ij} \\ \mu_{,i} &= \frac{2R}{n(n-1)} \lambda_i \end{aligned}$$

relatively to tensor a_{ij}, vector λ_i and invariant μ.

Theorem 2. Quasi-Einstein pseudo-Riemannian space with constant scalar curvature is closed relatively to nontrivial geodesic mappings.

Some systems of nonlinear PDE which are soluble in closed form

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The goal of this lecture is to study so-called Riemannian manifolds of conullity two. This means that, at any point, there is an orthonormal basis such that the each curvature component with at least free distinct indices is always equal to zero. Most "geometric classes" of such manifolds in dimension 3 can be expressed in an explicit form, using only arithmetic operations, differentiation and integration, involving some number of arbitrary functions.

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The gap phenomenon in parabolic geometry

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Abstract: In 2014 together with Dennis The we resolved the gap problem in complex or split-real parabolic geometry, i.e. we computed the amount of submaximal symmetry for every geometry in the class. Results of this type have been known for selected geometries since Ricci, Tresse, Fubini, Cartan, Egorov, Kobayashi, Sinyukov, Yano and some others via specific techniques. However it was in our paper that we first presented a universal solution for a large class of geometries, including conformal structures, systems of second order ODE, almost Grassmanian and Lagrangian structures, generic parabolic distributions, exceptional geometries etc. In later development we covered CR-structures, c-projective structures and some other real (non-split) specification. I will review the results and overview further developments and problems.

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About concircular infinitesimal transformations in the second approximation Riemannian space

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For a given Riemannian space $V_n(x; g)$ in the surrounding area of its random fixed rpoint $M_0(x_0^h)$ we have built the associated invariant space of the second approximation $\widetilde{V}_n^2(y; \widetilde{g}(y))$ ([2],[3]):

$$\widetilde{g}_{ij}(y) = \underset{\circ}{\mathrm{g}}_{ij} + \frac{1}{3} \underset{\circ}{\mathrm{R}}_{i\alpha\beta j}\, y^\alpha y^\beta, \tag{1}$$

where $\mathrm{g}_{ij_\circ} = g_{ij}(M_0)$, $\underset{\circ}{\mathrm{R}}_{i\alpha\beta j} = R_{i\alpha\beta j}(M_0)$.

In the space $\widetilde{V}_n^2$ we've examined concircular infinitesimal transformations ([4])

$$y'^h = y^h + \widetilde{\xi}^h(y)\delta t \tag{2}$$

While we were examining the first group of system equations:

$$\begin{array}{l} \widetilde{\nabla}_{(i}\widetilde{\xi}_{j)} = \psi(y)\widetilde{g}_{ij} \\ \widetilde{\nabla}_{ij}\psi = \phi(y)\widetilde{g}_{ij}, \end{array} \tag{3}$$

we have got an expression of the displacement vector $\widetilde{\xi}^h(y)$ of this transformations in the form of a uniformly convergent power series:

$$\begin{gathered} \widetilde{\xi}^h(y) = a^h_{.} + a^h_{.l}y^l + a^\alpha t^h_\alpha + \frac{1}{2}\left(\underset{1}{\mathrm{b}}\, y^h - \frac{1}{2}Ab^h\right) + \\ + \sum_{p=2}^{\infty}\left[\frac{(-1)^{p+1}}{2p-1}a^\alpha t^{(p)h}_{\ \ \alpha} + \frac{1}{2p}\left(\underset{2p-1}{\mathrm{b}}\, y^h - \frac{1}{2}A\underset{2p-2}{\mathrm{b}^h}\right) + \right. \\ \left. + \frac{(-1)^p}{4(2p-1)}A\sum_{s=1}^{p-1}\frac{2p-2s-1}{p-s}\underset{2p-2s-2}{\mathrm{b}^\alpha}\, t^{(s)h}_{\ \ \alpha}\right] + \frac{1}{3}\left(\underset{2}{\mathrm{b}}\, y^h - \frac{1}{2}A\underset{1}{\mathrm{b}^h}\right) + \\ + \frac{1}{12}A\underset{1}{\mathrm{b}^\alpha}\, t^h_\alpha + \frac{1}{5}\left(\underset{4}{\mathrm{b}}\, y^h - \frac{1}{2}A\underset{3}{\mathrm{b}^h}\right) + \\ + \sum_{p=3}^{\infty}\left\{\frac{1}{2p+1}\left(\underset{2p}{\mathrm{b}}\, y^h - \frac{1}{2}A\underset{2p-1}{\mathrm{b}^h}\right) + \frac{A}{2p}\left[\frac{p+1}{2p+1}\underset{2p-3}{\mathrm{b}^\alpha}\, t^h_\alpha + \right.\right. \\ \left.\left. + \sum_{s=2}^{p-1}\frac{(-1)^{s+1}(p-s)}{2p-2s+1}\underset{2p-2s+1}{\mathrm{b}^\alpha}\, t^{(s)h}_{\ \ \alpha}\right]\right\}, \end{gathered} \tag{4}$$

where $A = \underset{\circ}{\mathrm{g}}_{l_1 l_2}\, y^{l_1}y^{l_2}$, $t^h_p = \dfrac{1}{3}\underset{\circ}{\mathrm{R}}^h_{.l_1 l_2 p}\, y^{l_1}y^{l_2}$, $t^{(p)h}_{\ \ s} = t^{(p-1)h}_{\ \ \ \ \alpha}t^\alpha_s \quad (p = 2, 3, ...)$.

Then while we were examining the second group of system equations (3), we have got an expression of function $\psi(y)$ through the function $\phi(y)$ and objects of the space V_n in the point M_0:

$$\underset{p+1}{\mathrm{b}} = \frac{1}{p(p+1)}A\underset{p-1}{\mathrm{C}} \quad (p = 1, 2, ...), \tag{5}$$

where $\psi(y) = b + \sum\limits_{p=1}^{\infty} b_p, \phi(y) = c + \sum\limits_{p=1}^{\infty} c_p$,

and $\underset{p}{\mathrm{b}} = b_{l_1 \ldots l_p}y^{l_1}\cdot \ldots \cdot y^{l_p}$, $\underset{p}{\mathrm{c}} = c_{l_1 \ldots l_p}y^{l_1}\cdot \ldots \cdot y^{l_p}$, and $b_{l_1 \ldots l_p}, c_{l_1 \ldots l_p}$ are constants.

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Problem with integral conditions for evolution equations in Banach space

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Let A be a given linear operator acting in the Banach space B, and for this operator, arbitrary powers $A^n : B \to B$, $n \in \mathbb{N}$. Denote be $x(\lambda)$ the eigenvector of the operator A which corresponds to its eigenvalue $\lambda \in \Lambda$, i.s. nonzero solution in B of the equation $Ax(\lambda) = \lambda x(\lambda), \lambda \in \Lambda$, where $\lambda \subset \mathbb{C}$. If Λ is not an eigenvalue of the operator A then $x(\lambda) = 0$.

We consider next problem with integrals condition

$$\frac{d^2u}{dt^2} + a(A)\frac{du}{dt} + b(A)u = 0, \quad t \in [0, T], \tag{1}$$

$$\int_0^\alpha u(t)dt + \int_\beta^h u(t)dt = \varphi_1, \quad \int_0^\alpha tu(t)dt + \int_\beta^h tu(t)dt = \varphi_2, \tag{2}$$

where $\varphi_1, \varphi_2 \in B$, $\alpha > 0$, $\beta > 0$, $h > 0$, $\alpha < \beta < h < \infty$, $u : (0;\alpha) \cup (\beta; h) \to B$ - is an unknown function, $a(A) : B \to B$, $b(A) : B \to B$ - is abstract operators with entire symbols $a(\lambda) \neq const$, $b(\lambda) \neq const$.

Let for $m = \{0, 1\}$ function $M_m(t, \lambda)$ be a solution of the problem

$$\frac{d^2M_m(t,\lambda)}{dt^2} + a(\lambda)\frac{dM_m(t,\lambda)}{dt} + b(\lambda)M_m(t,\lambda) = 0, \quad t \in [0, T], \tag{3}$$

$$\int_0^\alpha t^k M_m(t,\lambda)dt + \int_\beta^h t^k M_m(t,\lambda)dt = \delta_{km}, \quad k = \{0, 1\}, \tag{4}$$

where δ_{km} is the Kronecker symbol.

Definition. We shall say that vectors $\varphi_1, \varphi_2 \in B$, from B belong $L \subset B$. If dependent exists on linear operators $R_{\varphi_k}(\lambda) : B \to B$, $\lambda \in \Lambda$ and measures μ_{φ_k} such that

$$\varphi_k = \int_\Lambda R_{\varphi_k}(\lambda)x(\lambda)d\mu_{\varphi_k(\lambda)}. \tag{5}$$

Theorem. Let in the problem (1), (2), the vectors φ_k belongs L. There $\varphi_k, k = \{1, 2\}$ can be represented in the form (5). Then the formula

$$U(t) = \int_\Lambda R_{\varphi_1}(\lambda)\{M_0(t,\lambda)x(\lambda)d\mu_{\varphi_1}(\lambda) + \int_\Lambda R_{\varphi_2}(\lambda)\{M_1(t,\lambda)x(\lambda)d\mu_{\varphi_2}(\lambda),$$

defines solution of the problem (1), (2), $M_m(t,\lambda)$ is a solution of the problem (3), (4).

Be means of the differential-symbol method [1] we construct of the problem (1), (2).

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Some aspects of the theory of infinitely small almost geodesic transformations of affinely connected spaces with torsion

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Affinely connected spaces A^n of class of smooth C^r $(n > 2,\ r > 1)$ are considered. As it is known [1], a curve L is called an almost geodesic line of the space A^n, if in every point of the curve coplanar along L two-way distribution, that contains a tangent vector of this curve, exists.

An infinitely-small transformation

$$\widetilde{x}^h = x^h + \varepsilon\xi^h(x^1; x^2; ...; x^n)$$

of an affinely connected space A^n is called almost geodesic, if as a result of it every geodesic line of the space A^n turns into a curve, that in main part, i.e. with neglecting by the addends of the second and higher order of smallness relative to the parameter ε, is an almost geodesic line of the space A^n.

Special almost geodesic transformations of the type $\Pi_2^4(\xi;\mu)$ of affinely connected spaces A^n with torsion are considered.

It is proved, that, in case of analytical character of vector fields ξ^h relative to μ_i^h structure, a Lie group $\widetilde{\Pi}_{\underset{\sim}{2}}^4(\xi;\mu)$ of infinitely small transformations is defined in affinely connected space $\widetilde{A}^n$ without torsion, that is associated with the space A^n.

The maximal order of the group is found. It is shown that, in particular, holomorphically-plane almost complex manifolds with an integrable structure present examples of spaces $\widetilde{A}^n$ of the maximal order mobility relative to the Lie group $\widetilde{\Pi}_2^4(\xi;\mu)$.

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On the homotopy types of right orbits of Morse functions on surfaces

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Let M be a connected orientable surface and $f : M \to P$ be a Morse map. Denote by $\mathcal{D}_{\mathrm{id}}$ the group of diffeomorphisms of M isotopic to the identity. This group acts from the right on the space of smooth maps $C^\infty(M,P)$ and one can define the stabilizer $\mathcal{S} = \{h \in \mathcal{D}_{\mathrm{id}} \mid f \circ h = f\}$ and the orbit $\mathcal{O} = \{f \circ h \mid h \in \mathcal{D}_{\mathrm{id}}\}$ of f with respect to that action.

On the other hand, the stabilizer $\mathcal{S}$ acts on the Kronrod-Reeb graph Γ of f. Denote by G the group of all automorphisms of Γ induced by elements from $\mathcal{S}$.

The homotopy type of $\mathcal{S}$ and the higher homotopy gruops of $\mathcal{O}$ were computed in [4]. In [5] it was proved that $\mathcal{O}$ has finite homotopy dimension, and in E. Kudryavtseva [2, 3] is was shown that $\mathcal{O}$ is homotopy equivalent to a quotient space of some p-dimensional torus by a free action of the above group G.

The aim of the talk is to describe precise algebraic structure of the fundamental group $\pi_1\mathcal{O}$ given in [6] for the case when M is distinct from the 2-sphere and 2-torus.

The case of 2-torus is considered in a series of papers by author and B. Feshchenko, see [7, 1]

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Foliations with all non-closed leaves on non-compact surfaces

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Let X be a 2-dimensional manifold possibly non-connected and having a boundary, and Δ be a one-dimensional foliation on X. We will say that Δ belongs to class $\mathcal{F}$ if it satisfies the following two conditions.

1. Every connected component ω of ∂X is a leaf of Δ.
2. Let $\omega \in \Delta$ be a leaf, and $J = [0, 1)$ if $\omega \subset \partial X$, and $J = (-1, 1)$ otherwise. Then there exists an open neighborhood U of ω and a homeomorphism $\phi : \mathbb{R} \times J \to U$ such that $\phi(\mathbb{R} \times 0) = \omega$ and $\phi(\mathbb{R} \times t)$ is a leaf of Δ for all $t \in J$.

Definition. Let X_i be a surface with a foliation Δ_i, $i = 1, 2$. Then a homeomorphism $h : X_1 \to X_2$ will be called *foliated* if it maps leaves of Δ_1 onto leaves of Δ_2.

Suppose Δ is a foliation of class $\mathcal{F}$ on a surface X. Let $Y = X/\Delta$ be the space of leaves, and $p : X \to Y$ be the corresponding quotient map. Endow Y with the quotient topology.

It follows from condition 2 above that each leaf of Δ is a closed subset of X, so Y is a T_1-space. However, in general, Y is not a Hausdorff space.

Definition. Let ω be a leaf of Δ and $y = p(\omega) \in Y$. We will say that ω is a *special* leaf and y is a *special* point of Y whenever Y is not Hausdorff at y.

Definition. A subset $S \subset \mathbb{R}^2$ will be called a *model strip* if there exist $a < b$ such that $\mathbb{R} \times (a, b) \subset S \subset \mathbb{R} \times [a, b]$ and the intersection $S \cap \mathbb{R} \times \{a, b\}$ is a disjoint union of open intervals.

A model strip $\mathbb{R} \times (a, b)$ will be called *open*.

Theorem. *Let X be a connected 2-dimensional manifold and Δ be a foliation on X belonging to class $\mathcal{F}$. Suppose that the family Σ of all special leaves of Δ is locally finite, and let Q be a connected component of $X \setminus (\Sigma \cup \partial X)$. Then the following statements hold true.*

1. *Q is foliated homeomorphic either with a standard cylinder C or a standard Möbius band M or an open model strip $\mathbb{R} \times (-1, 1)$. Moreover, in the first two cases $Q = X$.*

2. *Suppose Q is foliated homeomorphic with an open model strip. Fix any foliated homeomorphism $\phi : \mathbb{R} \times (-1,1) \to Q$ and denote*

$$A = \phi\big(\mathbb{R} \times (-1,0]\big), \qquad B = \phi\big(\mathbb{R} \times [0,1)\big).$$

Then the closures $\overline{A}$ and $\overline{B}$ are foliated homeomorphic to some model strips.

This theorem implies that the topological structure of the foliation $\Delta \in \mathcal{F}$ is uniquely determined by the combinatorics of gluing model strips.

Geometry of one infinitely symbolic representation of real numbers and metric problems associated with it

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Let $(q_j)_{j\in\mathbb{N}} \subset (0;1)$ be a sequence, and $M = ||m_{ik}||$ be the infinite matrix defined by $m_{ik} = q_{k-1}^{i-1}(1-q_{k-1})$. We will assume that $\prod_{n=1}^{\infty} m_{\alpha_n n} = 0$ for any sequence of positive integers (α_n). We consider a system of encoding (representation) of the fractional part of a real number x with infinite alphabet ($\alpha_n(x) = \alpha_n \in \mathbb{N}$) having a zero redundancy (any number has a unique representation) and depending on an infinitely many parameters q_j:

$$(0;1] \ni x = q_0^{\alpha_1} + \sum_{k=1}^{\infty} q_k^{\alpha_{k+1}} \prod_{n=1}^{k} q_{n-1}^{\alpha_n - 1}(1-q_{n-1}) \equiv \Delta_{\alpha_1\alpha_2\ldots\alpha_n\ldots}. \qquad (1)$$

The set $\Delta_{c_1c_2\ldots c_m} = \big\{x \mid x = \Delta_{c_1c_2\ldots c_m a_{m+1}a_{m+2}\ldots}, a_{m+i} \in \mathbb{N}, i \in \mathbb{N}\big\}$ will be called a *cylinder* of rank m. Our aim is to describe a geometry of the representation (1) (geometric meaning of digits, properties of cylidrical sets, etc.). The following simple properties of cylinders hold true:

1. $\bigcup\limits_{c_1=1}^{\infty} \bigcup\limits_{c_2=1}^{\infty} \ldots \bigcup\limits_{c_m=1}^{\infty} \Delta_{c_1c_2\ldots c_m} = (0;1]$; 2. $\Delta_{c_1c_2\ldots c_m} = \bigcup\limits_{i=1}^{\infty} \Delta_{c_1c_2\ldots c_m i}$;

3. $\Delta_{c_1\ldots c_m} = \big(\Delta_{c_1\ldots c_{m-1}[c_m+1]11\ldots}; \Delta_{c_1\ldots c_{m-1}c_m 11\ldots}\big]$;

4. The length of a cylinder: $|\Delta_{c_1\ldots c_m}| = \prod_{n=1}^{m} q_{n-1}^{c_n-1}(1-q_{n-1})$.

5. $|\Delta_{c_1\ldots c_m i}|/|\Delta_{c_1\ldots c_m}| = q_m^{i-1}(1-q_m) = m_{i(k+1)}$.

Lemma 1. *Let λ be the Lebesgue measure. Then*

1. $\lambda\left(\bigcup\limits_{i=k+1}^{\infty} \Delta_{c_1\ldots c_m i}\right) = (1-q_0)\ldots(1-q_{m-1})\, q_0^{c_1-1}\ldots q_{m-1}^{c_m-1} q_m^k$;

2. $\lambda\left(\bigcup\limits_{i=1}^{k} \Delta_{c_1\ldots c_m i}\right) = (1-q_0)\ldots(1-q_{m-1})\, q_0^{c_1-1}\ldots q_{m-1}^{c_m-1}\left(1-q_m^k\right)$;

3. $\lambda\left(\bigcup\limits_{i=k+1}^{n} \Delta_{c_1\ldots c_m i}\right) = (1-q_0)\ldots(1-q_{m-1})\, q_0^{c_1-1}\ldots q_{m-1}^{c_m-1} q_m^k\left(1-q_m^{n-k}\right)$.

Theorem 1. *Let (V_n) be a sequence of subsets of $\mathbb{N}$, and*

$$C = \{x \mid x = \Delta_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \alpha_n(x) \in V_n\}$$

1. C is a sum of half-intervals if $V_n = \mathbb{N}$ for all n larger than some n_0;
2. C is nowhere dense, if inequality $V_n \neq \mathbb{N}$ holds an infinite number of times;
3. $\lambda(C) = \prod_{k=1}^{\infty} \frac{\lambda(F_{k+1})}{\lambda(F_k)} = \prod_{k=1}^{\infty}\left(1 - \frac{\lambda(\bar{F}_{k+1})}{\lambda(F_k)}\right)$, where $F_0 = (0;1]$, $F_k = \bigcup\limits_{c_1\in V_1} \cdots \bigcup\limits_{c_{k-1}\in V_{k-1}} \bigcup\limits_{i\in V} \Delta_{c_1\ldots c_m i}$, $\bar{F}_{k+1} = F_k \setminus F_{k+1}$.

In particular, we give a negative answer to the following problem: suppose all V_n coincide with the same subset $V \subsetneq \mathbb{N}$. Is it true that $\lambda(C) = 0$? Set $q_j = \frac{1}{(j+2)^2}, j \in \mathbb{Z}_0$ and $V_n = \mathbb{N} \setminus \{i\}$ for some $i = 2, 3, \ldots$. Then $\lambda(C) > 0$.

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Topological stability of continuous functions with respect to averaging by measures with locally constant densities

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In [1] the authors obtained suffcient conditions for topological stability of continuous functions $f : \mathbb{R} \to \mathbb{R}$ having finitely many local extremes with respect to averagings. It is shown that this global problem reduces to a stability of germs of f near these local extremes.

In [1] the authors also obtained suffcient conditions for topological stability of germs with respect to averagings by discrete measures with finite supports.

In the present paper [2] we will give sufficient conditions for topological stability of germs with respect to measures with piece wise continuous (and in particular with piece wise constant) densities, see Theorems 1 and 2.

Theorem 1. *Let $f, g : [-\varepsilon; +\varepsilon] \to \mathbb{R}$ — be two piece wise 1– dierentiable functions and $h = f - g$. Suppose the following conditions hold:*

1) f and g strictly decrease on $[-\varepsilon; 0]$ and strictly increase on $[0; +\varepsilon]$;

2) there exists $C > 0$ such that for all $x \in [-\alpha; +\alpha]$ the following inequality holds:

$$f''_\alpha(x) \geq C\alpha;$$

3) the derivative $h' = g' - f'$ is continuous at 0 and $h'(0) = 0$. Then the germ of g at 0 is topologically stable with respect to averagings by measure μ.

Theorem 2. *Let $g : [-\varepsilon; \varepsilon] \to \mathbb{R}$ be a piece wise 1-differentiable function, satisfying the following conditions:*

1) g strictly decreases on $[-\varepsilon; 0]$ and strictly increases on $[0; +\varepsilon]$;

2) there exist finite limits

$$L = \lim_{x \to 0-0} g'(x), \qquad R = \lim_{x \to 0+0} g'(x).$$

For $i = 0; \dots ; n+1$ define the following numbers

$$Xi := L_\mu[t_0; t_i] + R_\mu[t_i; t_{n+1}] = L \sum_{j=0}^{i-1} (t_{j+1} - t_j)p_j + R \sum_{j=i-1}^{n} (t_{j+1} - t_j)p_j.$$

Suppose that for each $i \in 0; \dots ; n$ at least on of the numbers X_i and X_{i+1} is non-zero. Then the germ of g at 0 is topologically stable with respect to the averagings by measure μ.

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On ultrametric fractals generated by max-plus closed convex sets of idempotent measures

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In [3], the authors defined the notion of invariant idempotent measure on a complete ultrametric space. These measures are idempotent counterparts of the probabilistic fractals [2]. Recall that an idempotent measure on a compact Hausdorff space X is a functional $\mu\colon C(X) \to \mathbb{R}$ that preserves constants, the maximum operation and is weakly additive (i.e., preserves sums in which at least one summand is a constant function) [4]. Given an arbitrary metric space X, we denote by $I(X)$ the set of idempotent measures of compact supports on X.

Let (X, d) be an ultrametric space. Let us define an ultrametric on the set $I(X)$. For any $\varepsilon > 0$, denote by $\mathcal{F}_\varepsilon = \mathcal{F}_\varepsilon(X)$ the set of all functions $\varphi \in C(X)$ satisfying the property: for any $y \in \varphi(X)$ the set $\varphi^{-1}(y)$ is the union of open balls of radii ε. The space $I(X)$ of probability measures with compact supports is endowed with the metric $\hat{d}$:

$$\hat{d}(\mu, \nu) = \inf\{\varepsilon > 0 \mid \nu(\varphi) = \mu(\varphi), \text{ for every } \varphi \in \mathcal{F}_\varepsilon(X)\}.$$

This metric turns out to be an ultrametric on $I(X)$, see [1].

A nonempty subset $A \subset I(X)$ is called max-plus convex if $\max\{t+\mu, \nu\} \in A$ for every $\mu, \nu \in A$ and $t \in [-\infty, 0]$. We endow the set $ccI(X)$ of closed convex subsets of idempotent measures of compact support on X with the Hausdorff metric.

The aim of the talk is to obtain counterparts for the construction ccI of ultrametric fractals described in [3].

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The mean curvature flow associated to a density (paying special attention to curves)

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IвЂ™ll describe what is the mean curvature flow associated to a density (subject on which that I started to work with A. Borisenko) and will give some account of my recent work with F. Viñado-Lereu, divided in two parts:

In $\mathbb{R}^n$ with a density e^{ψ}, we study the mean curvature flow associated to the density (ψMCF) of a hypersurface. The main results of the first part concern with the description of the evolution under ψMCF of a closed embedded curve in the plane with a radial density, and with a statement of subconvergence to a ψ-minimal closed curve in a surface under some general circumstances.

In the second part we define Type I singularities for the ψMCF and describe the blow-up at singular time of these singularities. Special attention is paid to the case where the singularity come from the part of the ψ-curvature due to the density. We describe a family of curves whose evolution under ψMCF (in a Riemannian surface of non-negative curvature with a density which is singular at a geodesic of the surface) produces only type I singularities and study the limits of their blow-ups.

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Pseudosymmetric locally conformal Kaehler manifolds

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The notion of a locally conformal Kaehler manifold (an l.c.K-manifold) in a Hermitian Geometry has been introduced by I. Vaisman in 1976. In this work, we introduced the Walker type identities in l.c.K-space forms. Furthermore, the Roter type l.c.K-space forms are given. Moreover, the Bochner curvature tensor in l.c.K-manifolds and l.c.K-space forms are presented and some properties of the Bochner curvature tensor in an l.c.K-space form are studied.

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Minimal G-structures induced by the Lee form

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Let (M, g) be an oriented Riemannian manifold and denote by $SO(M)$ associated oriented orthonormal frame bundle. A G–structure on M is equivalent to existence of a reduction $P \subset SO(M)$ of a structure group $SO(n)$, $n = \dim M$, to G. Let ∇ be the Levi–Civita connection of g and denote by ω the connection form on the oriented orthonormal frame. Assume that on the level of Lie algebras the decomposition

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$$

is $\mathrm{ad}(G)$–invariant.

Introducing a natural Riemannian metric on $SO(M)$ we state the condition for minimality of P as a submanifold in $SO(M)$ [4]. It is an equation involving, so called, intrinsic torsion of a G–structure, which is a difference of the Levi–Civita connection ∇ and the G–connection ∇^G induced by the $\mathfrak{g}$–component $\omega_{\mathfrak{g}}$ of ω. We show that minimality is equivalent to harmonicity of the unique section σ_P of the associated bundle $SO(M)/G$. Here, harmonicity is considered as a map (not a section) with respect to some modification of a Riemannian metric on the base manifold M. This condition reminds condition of harmonicity of G–structures [1].

Finally, we give examples of minimal G–structures by examining condition of minimality in the case of locally conformally Kähler and contact metric structures. More precisely, we consider G–module $\mathcal{W}_4$ of the Gray–Hervella classes [3] of the intrinsic torsion in the case $G = U(n)$ and modules $\mathcal{C}_4$ and $\mathcal{C}_5$ in the case of $G = U(n) \times 1$ [2]. These structures can be characterized by the existence of the Lee form and equation of minimality is given by a certain condition on the vector field dual to the Lee form.

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Alexander Borisenko: 70 and counting

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I will present several results of Alexander Borisenko hand-picked from his vast and brilliant contribution to geometry (so far!)

Generalized helices in three dimensional Lie groups

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We *define three types* of helices on 3-dimensional Lie group with *left-invariant* metric as follows.

Definition 1. *Let G^3 be Lie group with left-invariant metric. Denote by $<\cdot,\cdot>$ the corresponding scalar product. Let γ be a parameterized curve with the Frenet frame T, N and B. The curve γ is called generalized helix of the first, second or third kind with axis ξ if there is a left-invariant along γ unit vector field ξ such that $< T, \xi >= const$, $< N, \xi >= const$ or $< B, \xi >= const$, respectively.*

For a given curve, we introduce a Frenet-type frame (τ, ν, β);curvature k_0 and torsion $\varkappa_0$ appeared in Frenet-type formulas; a group-curvature $k_G =| \mu(T)\times T |$ and a group-torsion $\varkappa_G =| \mu(T) \times B |$ of the curve, where $\mu(T)$ is affine transformation defined by the connection coefficients of the group.

The relations between the introduced above functions has the following form $k_G^2 = (k - k_0)^2 + 4kk_0 \sin^2(\alpha/2)$, $\quad \varkappa_G^2 = k_0^2 \sin\alpha^2 + (\varkappa - \varkappa_0 + \dot{\alpha})^2$, where $\alpha = \alpha(s)$ is the angle between N and dot-principal normal ν, $\dot{\alpha} = \frac{d\alpha}{ds}$. The results are the following ones.

Theorem 1. *Let γ be a parameterized curve in 3-dimensional Lie group G with left-invariant metric. Then γ is a helix of the first kind if and only if $\frac{\varkappa_0}{k_0} = \cot\theta$, where $\cos\theta = \langle T, \xi\rangle$.*

Theorem 2. *Let γ be a parameterized curve in 3-dimensional Lie group G with a left-invariant metric. Then γ is a helix of the second kind if and only if $\frac{k_0\cos\alpha(H^2+1)^{\frac{3}{2}}}{\dot{H}-k_0\sin\alpha(H^2+1)} = \tan\theta$, where $\cos\theta = \langle N, \xi\rangle$ and $H = \dfrac{\varkappa_0 - \dot{\alpha}}{k_0\cos\alpha}$.*

Theorem 3. *Let γ be a parameterized curve in 3-dimensional Lie group G with a left-invariant metric. Then γ is a helix of the third kind if and only if $\frac{k_0\sin\alpha(Q^2+1)^{\frac{3}{2}}}{\dot{Q}-k_0\cos\alpha(Q^2+1)} = \tan\theta$, where $\cos\theta = \langle B, \xi\rangle$ and $Q = \dfrac{\dot{\alpha} - \varkappa_0}{k_0\sin\alpha} = -\cot\alpha H$.*

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Curvature properties of statistical structures

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By a statistical structure on a manifold M we mean a pair (g, ∇), where g is a metric tensor field and ∇ is a torsion-free connection on M for which ∇g as a $(0,3)$-tensor field is symmetric in all arguments. The structures naturally appear in affine differential geometry, the theory of Lagrangian submanifolds, the theory of the second fundamental form, statistics and information theory. To each statistical structure a few curvature tensors can be attributed. At least three independent sectional curvatures can be built out of the curvature tensors. On the base of these concepts one can prove various theorems, both of local and global type.

On 4-dimensional Golden-Walker structures

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In this work, we show a method of pure metrics construction on a semi-Riemannian 4-manifold of neutral signature with respect to Golden structures. As an illustration, by applying the method, we exhibit explicitly pure metrics on Walker 4-manifolds. Moreover, we present some examples for 4-dimensional Golden-Walker structures.

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H-contact unit tangent sphere bundles

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The *energy* $E(V)$ of a unit vector field V on a compact, orientable Riemannian manifold $(\tilde{M}, \tilde{g})$ is defined as the energy of the corresponding map between $(\tilde{M}, \tilde{g})$ and its tangent sphere bundle equipped with the Sasaki metric: $E(V) = \frac{1}{2}\int_M |dV|^2 dv_{\tilde{g}}$. Critical points of E are called *harmonic* vector fields [1]; by considering the first variation one gets a local condition for harmonicity which requires neither compactness, nor orientability.

A contact metric manifold whose characteristic vector field is harmonic is called *H-contact*. The study of H-contact manifolds attracted considerable interest in the case when the contact metric manifold is itself the unit tangent sphere bundle of a Riemannian manifold (M^n, g) equipped with the Sasaki metric and the standard contact structure. For $n = 2, 3$, such an (M^n, g) must be of constant curvature [2]; the same is true if it is conformally flat [3], where local characterisation in the general case is also obtained; (M^n, g) is 2-stein, provided it is either Einstein [5] or $n = 4$ [6]. We prove the following.

Theorem. *Let (M, g) be a Riemannian manifold. The unit tangent sphere bundle T_1M equipped with the standard contact metric structure is H-contact if and only if (M, g) is 2-stein.*

A Riemannian manifold (M^n, g) is called *2-stein* if there exist two functions $f_1, f_2 : M \to \mathbb{R}$ such that for every $p \in M$ and every vector $X \in T_pM$,

$$\mathrm{Tr} R_X = f_1(p)\|X\|^2, \qquad \mathrm{Tr}(R_X^2) = f_2(p)\|X\|^4,$$

where R_X is the Jacobi operator [4]. 2-stein manifolds are classified in dimension $n \leq 5$, in locally symmetric case and in some other cases.

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Topology of functions and flows on low-dimensional manifolds with the boundary

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We consider the functions and flows on 2 and 3-dimensional manifolds with the boundary, all critical (fixed) points of which belong to the boundary of the manyfold. In this case there is the analogue of Morse functions. They are functions which have only non-degenerated critical points and their restrictions to the boundary have the same critical points that are also non-degenerated. There is the neighborhood of each of these points in such a way that the function f takes one of the following forms: $f(x,y) = -x_1^2 - ... - x_i^2 + x_{i+1}^2 + ... + x_{n-1}^2 \pm x_n, x_n \geq 0$, [1]. Besides in the case of isolated singular points on 2-manifold, the function can be represented in the form $f(x,y) = Re(x+iy), y \geq 0$ for some appropriate local coordinates (x,y).

Gradient-like flows of Morse functions in general position are Morse-Smale flows without orbits. On manifolds M with boundary ∂M it is a flow X which satisfies the following conditions:

1) the set of nonwandering points $\Omega(X)$ has finite number of orbits and all of them are hyperbolic,

2) if $u, v \in \Omega(X) \cap \mathrm{Int}M$ then unstable manifold $W^u(u)$ is transversal to stable manifold $W^s(v)$,

3) for $u, v \in \Omega(X)$, if $x \in M$ is a point of nontransversal intersection of $W^u(u)$ with $W^s(v)$ then $x \in \partial M$ and either u or v is a singularity of X [2].

Morse-Smale flows on the surface with boundary can have four types of fixed points on the boundary: 1) a source, 2) a sink, 3) a-saddle and 4) b-saddle. The topological structure of such flows is determined by the separatrixes.

There is a flow with one singular point for any connected surface with a connected boundary. Separatrix breaks neighborhood of this point into the corners that can have four types: 1) hyperbolic, 2) eliptic 3) sources and 4) sink. Location separatrix and specifying types of angles determines the structure of such flows.

In dimension 3 generalized Heegaard diagrams [3] can be used to determine the structure of Morse-Smale flows.

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Caustic of space curve

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The following two problems are well known for students: (1) for a given curve on a plane, find a caustic (the envelope) of rays emitted by the curve in a direction of its normal vector field; (2) for a given curve on a plane and a given luminous flux, find a caustic of the reflected luminous flux. The first problem solves the evolute; the second one easily can be solved by using the Frenet formulas.

Suppose γ be a space curve. Is it possible to find a caustic of rays emitted by the curve? Evidently, the caustic can be formed by some specific bunch of rays. Namely, the rays should form a **developable surface** and the caustic is nothing else but **stiction line** on it.

Generalization of the problem (2) looks senseless because for a space curve there is no definite reflection low. On the other hand, if a curve is located on a surface, then the reflection law is well-known.

We pose the following problem: **for a given space curve, find (construct) a surface bend in such a way that the reflected bunch of rays would have a caustic.** We solve this problem and give graphical presentation of the solution.

Topologically equivalent singular sesquilinear forms

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This is a joint work with C.M. da Fonseca and V.V. Sergeichuk.

Two sesquilinear forms $\Phi : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}$ and $\Psi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ are called *topologically equivalent* if there exists a homeomorphism $\varphi : \mathbb{C}^m \to \mathbb{C}^n$ (i.e., a continuous bijection whose inverse is also a continuous bijection) such that

$$\Phi(x, y) = \Psi(\varphi(x), \varphi(y)) \quad \text{for all } \ x, y \in \mathbb{C}^m.$$

R.A. Horn and V.V. Sergeichuk in [1] constructed a *regularizing decomposition* of a square complex matrix A; that is, a direct sum

$$SAS^* = R \oplus J_{n_1} \oplus \cdots \oplus J_{n_p},$$

in which S and R are nonsingular and each J_{n_i} is the n_i-by-n_i singular Jordan block.

In [2] we prove that two sesquilinear forms Φ and Ψ are topologically equivalent if and only if the regularizing decompositions of their matrices coincide up to permutation of the singular summands J_{n_i} and replacement of $R \in \mathbb{C}^{r\times r}$ by a nonsingular matrix $R' \in \mathbb{C}^{r\times r}$ such that R and R' are the matrices of topologically equivalent forms $\mathbb{C}^r \times \mathbb{C}^r \to \mathbb{C}$.

Analogous results for bilinear forms over $\mathbb{C}$ and over $\mathbb{R}$ are also obtained.

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Manifolds and surfaces with locally Euclidean metrics

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The metric of a Riemannian manifold M^n is locally Euclidean (l.E.) if for any point of M^n there exists a neighborhood isometric to a ball with the standart Euclidean metric.

For isometric immersions of such a metric there is a special case concerning its immersion in the standart Euclidean n-space. We study the case $n = 2$ and give some necessary/sufficient conditions for possibility of such immersions. If the two-dimensional domain D with a l.E. metric is multi-connected then there are cases when it is not isometrically immersiable in R^2 and then one can distinct the cases of cylindrical and conical singularities in the behavior of the metric. As to an immersion in three-space we prove that any l.E. metric immersable in R^2 is embeddable in R^3 (see [1]).

Further we study the ruled surfaces with I.E. metrics in the smallest class of smoothness C^1 and give a complete description of their structure [2].

The following interesting question is the local and global behavior of solutions of the trivial Monge-Ampère equation $z_{xx}z_{yy} - z_{xy}^2 = 0$ (1). We prove some theorems on the regularities of its solutions and formulate a problem on the description of structure of its isolated singularities. For the last question we prove that for any set A with a finite number of points the equation (1) has a C^∞-smooth solution defined over the whole plane (x, y) and having isolated non-removable singularities just at points in the set A. For some special cases the set of singular points can consist from discrete infinite number of points (these results are not yet published).

A part of results concerning the singular points of solutions to the equation (1) have some intersections with ones obtained independently by J. Gálvez and B. Nelli too [3].

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A different angle definition in non-Euclidean spaces

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In this study, we examine an angle definition for the geometric objects, which have null-edges, by using specail transformation for two dimensional non-Euclidean spaces. Some trigonometric laws obtained by [1, 2, 3] for non-null objects in non-Euclidean spaces. We extend similiar laws for null objects in two dimensional non-Euclidean spaces.

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The estimate from above for self-perimeter of a unit circle by its diameter on the Minkowski plane

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Let A^2 be affine plane. In what follows, we identify the points of A^2 with their position-vectors. Let B be convex compact figure on A^2 containing the origin O inside. Each pair $(B;O)$ defines uniquely the distance function.

The distance function $g_B(x)$ defines the distance between arbitrary points x and y on A^2 by $\rho_B(x;y) = g_B(y-x)$.

Affine plane A^2 with the metric ρ_B is called the Minkowski plane M^2. The point O is called the origin on M^2. The figure B is called the normalizing figure or the unit circle on M^2 (see [1]). Denote by $L^+(B)$ the length of ∂B anti-clockwise and by $L^-(B)$ the length of ∂B clockwise. S. Golab [2] proved that if B is centrally symmetric with respect to the origin O, then $L(B) = L^-(B) = L^+(B)$ and the following sharp estimates $6 \le L(B) \le 8$ hold.

Very simple examples show that there is no absolute constant which bounds from above the self-perimeters $L^{\mp}(B)$ for non-symmetric normalizing figure.

The value $D(B) = \max\limits_{x,y\in B} \rho(x;y)$ is called diameter of the normalizing figure B on M^2. In present paper we give estimates from above on the self-perimeters $L^{\mp}(B)$ by the self-diameter $D = D(B)$ of a unit circle B on M^2 (see [3]).

Theorem. If P_4 is a normalizing quadrangle of diameter $D = D(P_4)$, then

$$L^{\mp}(P_4) \le \frac{2D^2}{D-1}.$$

The estimate is sharp.

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Some aspects of geometry of tangent bundle induced by invariant approximations of the base affinely connected space

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Consideration of Riemannian coordinate system with the beginning in any fixed point in a real affinely connected torsion free space with the base manifold X^n admits to receive an invariant series of Tailor's type for any tensor and for the object of affine connection of the space. The coefficients of the series depends not only from coordinates of the current point but also from components of tangent element in it. We reject the addends of the second and higher order of smallness relative to the components y^n of tangent element in the series for the components $\Gamma^h_{ij}(x)$ of the object of affine connection of the space A^n. As a result we obtain the next components of the object of affine connection:

$$\widetilde{\Gamma}^h_{ij}(x;y) = \Gamma^h_{ij}(x) - \frac{1}{3} R^h_{(ij)\alpha}(x) y^\alpha. \tag{1}$$

These components define on X^n a geometry that is in a natural way connected with the invariant theory of approximations in affinely connected spaces A^n [1]. The affine connection $\widetilde{\Gamma}$ on X^n is considered as a broadening of affine connection Γ. In some sense it is similar to connections of Cartan and Bervald of Finsler geometry.

For tensor fields on manifold X^n, those depend not only from the coordinates of current point, but also from the components of tangent element in it, we introduce the covariant differentiation by the rule:

$$T^h_i(x;y)_{;j} = \frac{\partial T^h_i}{\partial x^j} - y^\alpha \widetilde{\Gamma}^\beta_{\alpha j} \frac{\partial T^h_i}{\partial y^\beta} + \widetilde{\Gamma}^h_{j\alpha} T^\alpha_i - \widetilde{\Gamma}^\alpha_{ji} T^h_\alpha. \tag{2}$$

The connection (1) can be broaden on the tangent bundle $T(A^n)$ by natural way for example by the rule of complete lift. As a result, possibility of development geometry of tangent bundles $T(A^n)$, in a natural way connected with a theory of approximations in affinaly connected spaces A^n, appears.

Some geometric properties of tangent bundle $T(A^n)$ are investigated. Among them, in particular, there are geometric senses of vanishing of Riemann and Ricci tensors if the last ones are found with the help of the broaden affine connection (1), and geometric senses of vanishing of the first covariant derivatives of Riemann and Ricci tensors in cases the derivatives are obtained in correspondence with the law (2).

The question of existence of geodesic mappings of the spaces $T(A^n)$ are also considered.

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Homeotopy groups of rooted tree like non-singular foliations

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A subset $S \subset \mathbb{R}\times[0;1]$ will be called *a model strip* if the following conditions hold: 1) $\mathbb{R}\times(0;1) \subseteq S$; 2) $\partial_- S = S \cap \mathbb{R}\times\{0\} \cong (0,1)$; 3) $\partial_+ S = S \cap \mathbb{R}\times\{1\}$ is an union of open finite intervals and the closures of intervals in $\partial_+ S$ constitute a locally finite family in $\mathbb{R}^2$. By *a standard gluing* of model strip S_2 to a model strip S_1 we will mean the gluing of lower boundary $\partial_- S_2$ to some of intervals of $\partial_+ S_1$ by the preserving orientation affine homeomorphism.

Every model strip S will be called *a stripped surface* Σ *of diameter 0* and the boundary intervals of $\partial_+ S$ will be named *the boundary intervals of* Σ *of level 1.* By induction, if a stripped surface Σ of diameter i, $i \geq 0$ and the set of boundary intervals of level $i+1$ are defined, then by *a stripped surface of diameter* $i+1$ we will mean the surface obtained by the standard gluing of some family of model strips $\{S_\lambda\}_{\lambda\in\Lambda}$ to some of boundary intervals of Σ of level $i+1$. In this case the boundary intervals of $\bigsqcup_{\lambda\in\Lambda} \partial_+ S_\lambda$ will be called *the boundary intervals of level* $i+2$. Denote by $\mathfrak{F}$ the class of stripped surfaces of a finite diameter.

Notice that every model strip has an oriented foliation consisting of horizontal lines $\mathbb{R}\times t$, $t \in (0,1)$ and connected components of ∂S. Since a standard gluing identifies leaves of such foliations, we see that every stripped surface has the foliation F consisting of oriented foliations on model strips. This foliation will be called *canonical*.

Let $H^+(F)$ be the group of homomorphisms of $\Sigma \in \mathfrak{F}$ which maps leaves onto leaves preserving their orientation and $H_0^+(F)$ be the identity path component of $H^+(F)$. A quotient-group $\pi_0 H^+(F) = H^+(F)/H_0^+(F)$ will be called the homeotopy group of F. The algebraic structure of $\pi_0 H^+(F)$ was described in [1].

Denote by $G(F)$ the space of leaves $\Sigma\diagup F$. It can be regarded as "non-Hausdorff" graph which has "split" vertices. Let $H(G)$ be the group of all homeomorphisms of $G(F)$. It is easy to see that each homeomorphism $h \in H^+(F)$ induces the homeomorphism $\rho(h) : G(F) \to G(F)$. So, that the correspondence $h \mapsto \rho(h)$ is a homomorphism $\rho : H^+(F) \to H(G(F))$. Let $K = \rho(H^+(F))$ and K_0 be the group of homeomorphisms of $G(F)$ which isotopic to identity in K.

Theorem. Let $\Sigma \in \mathfrak{F}$ and F be a canonical foliation. Then $\pi_0 H^+(F)$ is isomorphic to $\pi_0 K = K/K_0$

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On the surfaces in Minkovski space which correspond to the stationary values of the sectional curvature of the Grassmann manifold

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The fact that the values of the sectional curvature of the Grassmann manifold of the Euclidean space belong to the segment $[0;2]$ has been proved in [1]. The surfaces with the minimal and maximal sectional curvature in the domains, which are tangential to the Grassmann image of the surface, have been studied in [2] and [3].

Let ${}^1R_4$ be Minkovski space (with metric $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$) and $PG(2,4)$ be the Grassmann manifold of the nonisotropic two-dimensional planes in ${}^1R_4$. The sectional curvature $K(\sigma)$ of this manifold can take on any real value. That is why we are going to analyze the values of the sectional curvature at points of the local extremum. From [4] we have the stationary values of the sectional curvature, which are equal 0 and 1.

Lets consider the classes of the regular surfaces V^2 in Minkovski space ${}^1R_4$ (spacelike or timelike) with the stationary values of the curvature of the Grassmann manifold $PG(2,4)$ along the domains, which are tangential to its the non-degenerated Grassmann image Γ^2.

Theorem 1. The stationary value of the sectional curvature of the Grassmann manifold $PG(2,4)$ along any nonisotropic domain, which is tangential to Γ^2 takes on 0 value if and only if V^2 is spacelike surface with zero Gauss curvature and flat normal connection.

Theorem 2. The stationary value of the sectional curvature of the Grassmann manifold $PG(2,4)$ along any nonisotropic domain, which is tangential to Γ^2 is equal to 1 if and only if either point codimension of the surface V^2 is equal to 1 or the Gauss curvature of the surface V^2 is equal to 0 and the coefficients of the second fundamental forms satisfy the following condition $L_{11}^1 L_{22}^2 + L_{11}^2 L_{22}^1 = 2L_{12}^1 L_{12}^2$.

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On the dynamics of non-invertible branched coverings of surfaces

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Let $f : M \to M$ be a branched covering, i.e. an inner map of a surface M. Recall that an inner map is an open and isolated map. A map is open if the image of an open set is open. A map is isolated if the pre-image of a point consists of isolated points.

The author introduced (see [1]) a set of new invariants of topological conjugacy of non-invertible inner mappings that are modeled from the invariant sets of dynamical systems generated by homeomorphisms. Those new invariants are based on the analogy between the trajectories of a homeomorphism and the directions in the set of points having common image which is viewed as having 2 dimensions.

In the talk we explore the dynamical properties of wandering sets of different classes of branched coverings.

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Integral formulae for foliations with singularities

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We shall discuss and provide conditions under which the integral formulae known ([1] – [5]) for codimension one foliations on closed manifolds hold for foliations defined on such manifolds outside a *set of singularities*, that is outside a union of pairwise disjoint closed submanifolds of codimension large enough. We shall discuss also some examples and applications.

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On anti-totally geodesic unit vector fields

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A unit vector ξ field on the Riemannian manifold M gives rise a (local) embedding $\xi : M \to T_1M$. By endowing T_1M with the Riemannian metric, we provide the image $\xi(M) \subset T_1M$ with intrinsic and intrinsic geometry. We can relate geometrical properties of the unit vector field with the geometrical properties of the submanifold $\xi(M)$. For instance, a unit vector field is said to be minimal or totally geodesic if the submanifold $\xi(M) \subset T_1M$ is minimal or totally geodesic, respectively.

Denote by ∇ the Levi-Civita connection on M and introduce the so-called Nomizu operator $A_\xi X = -\nabla_X \xi$ and denote by A^t_ξ its conjugate. Then the following mappings $\xi_* : TM \to T\big(\xi(M)\big)$ and $\tilde n : TM \to T^\perp\big(\xi(M)\big)$, namely,

$$\xi_* X = X^h - (A_\xi X)^{tg}, \qquad \tilde n(Y) = (A^t_\xi Y)^h + Y^{tg},$$

can be defined. Here $(\cdot)^h$ and $(\cdot)^{tg}$ mean the horizontal and tangential lifts into $T(T_1M)$, respectively. Introduce the tensor fields

$$Hess_\xi(X,Y) = \frac{1}{2}\big((\nabla_X A_\xi)Y + (\nabla_Y A_\xi)X\big)$$

and

$$\Gamma_\xi(X,Y) = \frac{1}{2}\big(R(A_\xi X,\xi)Y + R(A_\xi Y,\xi)X\big),$$

where $(\nabla_X A_\xi)Y = \nabla_X(A_\xi Y) - A_\xi(\nabla_X Y)$ and $R(\cdot,\cdot)\cdot$ is the curvature tensor of M. If we denote by $\tilde\nabla$ the Levi-Civita connection of the Sasaki metric on T_1M, then the Gauss decomposition for $\xi(M)$ can be expressed by

$$\tilde\nabla_{\xi_* X}\xi_* Y = \xi_*\Big(\nabla_X Y + \Gamma_\xi(X,Y)\Big) + \Big(A_\xi(\Gamma_\xi(X,Y)) - Hess_\xi(X,Y)\Big)^{tg}.$$

The $(\cdot)^{tg}$ component of the decomposition is transversal to $\xi(M)$ and its projection onto the normal bundle $T^{\perp}((\xi(M))$ defines the second fundamental form of $\xi(M) \subset T_1M$. If this projection is zero (for all X and Y), then the field ξ is called **totally geodesic**. We pose the question: *is there exists a unit vector field such that the transversal component of the Gauss decomposition above has zero projection onto $T(\xi(M))$?* Using the terminology from complex geometry we call such a vector field by **anti-totally geodesic** one.

The equation on anti-totally geodesic unit vector field takes the form

$$A^t_\xi\Big(A_\xi(\Gamma_\xi(X,Y)) - Hess_\xi(X,Y)\Big) = 0 \quad \textit{for all } X, Y.$$

The answer is YES. The example provides a unit invariant vector field on the Lie group $E(2)$ with the left-invariant metric.

On asymptotic dimension invariants

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The notion of decomposition complexity was introduced in [4] using a game theoretical approach. In [3], the authors introduced the notion of straight decomposition complexity. One of the aims of the talk is to prove that the property of of straight decomposition complexity is preserved by some constructions in the category of metric spaces.

Using a construction due to P. Borst [1, 2] we define the straight decomposition complexity degree $\mathrm{sDC}(X)$ of a metric space X. Some permanence properties of sDC are established.

In [6], the notion of the asymptotic power dimension is introduced. This notion is tightly related to the notion of subpower Higson corona [5]. We prove that this corona shares some properties with the Stone-Čech compactification of a proper metric space. In particular, this corona does not contain non-trivial convergent sequences.

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Open topological and geometrical problems in analysis

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Definition. We shall say that a family of sets $\mathfrak{J} = \{F_\alpha\}$ will assign the shade tangent to a manifold M in the point $x \in M$, if each straight line, tangent to variety M in the point , $x \in M \setminus \bigcup\limits_\alpha F_\alpha$ has a nonempty intersection with some set F_α from family $\mathfrak{J}$.

Problem Find the minimal number of balls, which two by two are not crossed, with the centre on sphere $S^2 \subset \mathbb{R}^3$, which will provide the shade tangent to the sphere S^2 in each point $x \in S^2 \setminus \bigcup\limits_\alpha F_\alpha$.

Theorem. *There exists the family from 14 open (closed) balls, which two by two are not crossed, with the centre on a sphere $S^2 \subset \mathbb{R}^3$, which will provide the shade tangent to the sphere S^2 in each point $x \in S^2 \setminus \bigcup\limits_{i=1}^{14} F_i$.*

The Opened questions. 1) What minimal number of balls solves the problem?

2) What minimal number of balls with the same radius solves the problem?

3) What is minimal family of balls $\mathfrak{J} = \{B_i\},\ i = 1, 2, \dots, m$, which two by two are not crossed, with the centre on a sphere $S^2 \subset \mathbb{R}^3$, will provide that each straight line getting through the arbitrary point $x \in B^3 \setminus \bigcup\limits_{i=1}^{m} B_i$, where B^3 be the closed ball bounded by the sphere S^2, will cross at least one of chosen balls?

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